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A VARIATIONAL FORMULATION AND INVESTIGATION OF BOUNDARY-VALUE PROBLEMS OF THE NON-LINEAR THEORY OF PLATES USING AN ENERGY APPROACH†

Ya. I. BURAK and G. I. MOROZ

Lviv, Ukraine

e-mail: burak@cmm.lviv.ua; moros@cmm.lviv.ua

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A variational formulation of boundary-value problems of the non-linear dynamic theory of elasticity using the Hamilton functional is presented. The quasi-static boundary-value problem for thin plates is considered. The initial system of equations, in a two-dimensional formulation, is represented in terms of generalized forces and displacements. The sufficient conditions for the existence and uniqueness of a weak solution are established. © 2004 Elsevier Ltd. All rights reserved.

1. THE VARIATIONAL FORMULATION OF BOUNDARY-VALUE PROBLEMS OF THE NON-LINEAR THEORY OF ELASTICITY

We will consider an elastically deformed isotropic solid body K^*. In the initial configuration the body is unloaded and, in Euclidean space, it uniformly fills the region X_0, bounded by the surface ∂X_0. The position of an arbitrary point $k \in K^*$ in the initial configuration ($\tau < \tau_1$) is characterized by the radius vector $\mathbf{r}_0$. In the time interval $[\tau_1, \tau_2]$ the body is acted upon by surface and body forces and, in the actual configuration ($\tau_1 \le \tau \le \tau_2$), occupies the region $X \cup \partial X$. The position of a point $k \in K^*$ at an arbitrary instant of time ($\tau_1 \le \tau \le \tau_2$) is defined by the radius vector $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, \tau)$.

In the variational formulation of the mathematical model of the non-linear theory of elasticity we take as the initial functional the Hamilton functional

$$F[\mathbf{u}, \boldsymbol{\nabla}_0 \otimes \mathbf{u}, \mathbf{p}] = \int_{\tau_1}^{\tau_2} \left\{ \int_{X_0} \left[H(\mathbf{p}, \boldsymbol{\nabla}_0 \otimes \mathbf{u}) + \mathbf{u} \cdot \left(\frac{\partial \mathbf{p}}{\partial \tau} - \mathbf{f}_0 \right) \right] dV_0 - \int_{\partial X_0} \boldsymbol{\sigma}_n^+ \cdot \mathbf{u} d\Sigma_0 \right\} d\tau - \int_{X_0} \mathbf{u}_{(2)}^* \cdot \mathbf{p}_{(2)} dV_0 \tag{1.1}$$

where $H = H(\mathbf{p}, \boldsymbol{\nabla}_0 \otimes \mathbf{u})$ is the Hamilton function, $\mathbf{p} = \int_{\tau_1}^{\tau_2} (\boldsymbol{\nabla}_0 \cdot \hat{\sigma} + \mathbf{f}_0) d\tilde{\tau}$ is the force momentum density vector, $\hat{\sigma}$ is the Piola–Kirchhoff stress tensor of the first kind, $\boldsymbol{\sigma}_n^+ = \boldsymbol{\sigma}_n^+(\mathbf{r}_0, \tau)$ is the vector of the surface forces, $\mathbf{f}_0 = \mathbf{f}_0(\mathbf{r}_0, \tau)$ is the vector of the mass-force density acting in the region X_0, $\mathbf{u} = \mathbf{r} - \mathbf{r}_0$ is the vector of the displacement of a point of the body from the initial configuration to the actual configuration, $\boldsymbol{\upsilon} = \partial \mathbf{u}/\partial \tau$ is the velocity vector, $\boldsymbol{\nabla}_0 \equiv \partial/\partial \mathbf{r}_0$ is the Hamilton operator, $\boldsymbol{\nabla}_0 \otimes \mathbf{u} = \boldsymbol{\nabla}_0 \otimes \mathbf{r} - \hat{I}$, $\boldsymbol{\nabla}_0 \otimes \mathbf{r}$ is the tensor of the gradient of the location, $\hat{I}$ is the unit tensor, $\mathbf{u}_{(2)}^*(\mathbf{r}_0) = \mathbf{u}(\mathbf{r}_0, \tau_2)$ is the specified field of the displacement vector at the instant of time τ_2 and $\mathbf{p}_{(2)}(\mathbf{r}_0) \equiv \mathbf{p}(\mathbf{r}_0, \tau_2)$.

Here and henceforth all the additive parameters of the physically small region $\delta K \subset K^*$ are normalized with respect to the geometrical parameters δV_0, $\delta \Sigma_0$ of this region in the initial state. In particular

$$\delta H = H \delta V_0, \quad \delta \mathbf{P} = \mathbf{p} \delta V_0, \quad \boldsymbol{\sigma}_n^+ = \boldsymbol{\sigma}_n^{+*} \frac{d\Sigma^*}{d\Sigma_0}, \quad \mathbf{f}_0 = \mathbf{f}^* \frac{dV^*}{dV_0}$$

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where H and $\mathbf{p}$ are the densities of the additive parameters δH and $\delta \mathbf{P}$, $\boldsymbol{\sigma}_n^{+*}$ and $\mathbf{f}^*$ are the vectors of the surface forces and mass forces in the actual configuration dV^*, and $d\Sigma^*$ are the geometrical parameters of a physically small region in the actual configuration.

The necessary condition for a minimum of the Hamilton functional is that its first variation

$$\delta F[\mathbf{u}, \nabla_0 \otimes \mathbf{u}, \mathbf{p}] =$$

$$= \int_{\tau_1}^{\tau_2} \left\{ \int_{X_0} \left[\left(\frac{\partial H}{\partial \mathbf{p}} - \boldsymbol{\upsilon} \right) \cdot \delta \mathbf{p} + \left(\frac{\partial H}{\partial (\nabla_0 \otimes \mathbf{u})} - \hat{\sigma} \right) \cdot\cdot\, \delta (\nabla_0 \otimes \mathbf{u})^T \right] dV_0 + \int_{\partial X_0} (\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^+) \cdot \delta \mathbf{u} d\Sigma_0 \right\} d\tau + \tag{1.2}$$

$$+ \int_{X_0} [(\mathbf{u}_{(2)} - \mathbf{u}^*_{(2)}) \cdot \delta \mathbf{p}_{(2)} - \mathbf{u}_{(1)} \cdot \delta \mathbf{p}_{(1)}] dV_0 = 0$$

should be equal to zero.

From the fact that the variations $\delta \mathbf{u}$, $\delta(\nabla_0 \otimes \mathbf{u})^T$, $\delta \mathbf{p}$ are independent we obtain the following governing relations of the model

$$\boldsymbol{\upsilon} = \frac{\partial H}{\partial \mathbf{p}} \equiv \boldsymbol{\upsilon}(\mathbf{p}, \nabla_0 \otimes \mathbf{u}), \quad \hat{\sigma} = \frac{\partial H}{\partial \nabla_0 \otimes \mathbf{u}} \equiv \hat{\sigma}(\mathbf{p}, \nabla_0 \otimes \mathbf{u}), \quad \mathbf{r}_0 \in X_0 \tag{1.3}$$

$$\boldsymbol{\sigma}_n \equiv \mathbf{n} \cdot \hat{\sigma} = \boldsymbol{\sigma}_n^+, \quad \mathbf{r}_0 \in \partial X_0 \tag{1.4}$$

$$\mathbf{u}|_{\tau = \tau_2} = \mathbf{u}^*_{(2)}, \quad \mathbf{r}_0 \in X_0 \tag{1.5}$$

It follows from relations (1.3) that the Hamilton function $H(\mathbf{p}, \nabla_0 \otimes \mathbf{u})$ is a function of the local state, Eqs (1.3)–(1.5) are the governing equations of the model, and, correspondingly, the differential 1-form

$$dH = \boldsymbol{\upsilon} \cdot d\mathbf{p} + \hat{\sigma} \cdot\cdot\, d(\nabla_0 \otimes \mathbf{u})^T \tag{1.6}$$

will be a total differential.

The sufficient condition for a minimum of the Hamilton functional (1.1) is the condition of its convexity

$$\delta^2 F = \int_{\tau_1}^{\tau_2} \left[\int_{\partial X_0} \delta \boldsymbol{\sigma}_n \cdot \delta \mathbf{u} d\Sigma_0 \right] d\tau + \int_{X_0} [\delta \mathbf{u}_{(2)} \cdot \delta \mathbf{p}_{(2)} - \delta \mathbf{u}_{(1)} \cdot \delta \mathbf{p}_{(1)}] dV_0 > 0 \tag{1.7}$$

which can be converted to the form

$$\delta^2 F = \int_{\tau_1}^{\tau_2} \left[\int_{X_0} \{ \delta \boldsymbol{\upsilon} \cdot \delta \mathbf{p} + \delta \hat{\sigma} \cdot\cdot\, \delta (\nabla_0 \otimes \mathbf{u})^T + 2(\nabla_0 \cdot \delta \hat{\sigma}) \cdot \delta \mathbf{u} \} dV_0 \right] d\tau =$$

$$= \int_{\tau_1}^{\tau_2} \int_{X_0} \{ \delta^2 H + 2(\nabla_0 \cdot \delta \hat{\sigma}) \cdot \delta \mathbf{u} \} \delta V_0 d\tau > 0$$

We take as the sufficient conditions for convexity, in particular, the conditions

$$\int_{\tau_1}^{\tau_2} \int_{X_0} \delta^2 H(\mathbf{p}, \nabla_0 \otimes \mathbf{u}) dV_0 d\tau > 0, \quad \int_{\tau_1}^{\tau_2} \int_{X_0} (\nabla_0 \cdot \delta \hat{\sigma}) \cdot \delta \mathbf{u} dV_0 d\tau \geq 0 \tag{1.8}$$

Relations (1.3)–(1.5) and (1.8) constitute the complete system of equations of the dynamic processes in elastic bodies.

To obtain the equations of motion, written in terms of the displacement vector, in functional (1.1) we must change from the Hamilton function $H(\mathbf{p}, \nabla_0 \otimes \mathbf{u})$ to the Lagrange function by a Legendre transformation [1]

$$L = \boldsymbol{\upsilon}\cdot\mathbf{p} - H \tag{1.9}$$

The sufficient condition for the function $L = L(\upsilon, \nabla_0 \otimes \mathbf{u})$ to exist is the satisfaction of the Legendre condition, which is equivalent to the condition

$$I_3\left(\frac{\partial\boldsymbol{\upsilon}}{\partial\mathbf{p}}\right) \neq 0; \quad I_3 = I\left(\frac{\partial\boldsymbol{\upsilon}}{\partial\mathbf{p}}\cdot\frac{\partial\boldsymbol{\upsilon}}{\partial\mathbf{p}}\cdot\frac{\partial\boldsymbol{\upsilon}}{\partial\mathbf{p}}\right) \equiv \left(\frac{\partial\boldsymbol{\upsilon}}{\partial\mathbf{p}}\cdot\frac{\partial\boldsymbol{\upsilon}}{\partial\mathbf{p}}\cdot\frac{\partial\boldsymbol{\upsilon}}{\partial\mathbf{p}}\right)\cdot\cdot\,\hat{I}$$

where I_3 is the third algebraic invariant of the tensor $\partial\boldsymbol{\upsilon}/\partial\mathbf{p}$.

The differential 1-form for the function L can be represented in the form

$$dL = \mathbf{p}\cdot d\boldsymbol{\upsilon} - \hat{\boldsymbol{\sigma}}\cdot\cdot\, d(\nabla_0\otimes\mathbf{u})^T$$

The function L is the function of state, specified in the phase space of the parameters $\boldsymbol{\upsilon}$, $\nabla_0 \otimes \mathbf{u}$. The conjugate to them will be the generalized forces $\mathbf{p}$ and $\hat{\boldsymbol{\sigma}}$, for which the following governing equations of state hold

$$\mathbf{p} = \frac{\partial L}{\partial\boldsymbol{\upsilon}} \equiv \mathbf{p}(\boldsymbol{\upsilon}, \nabla_0\otimes\mathbf{u}), \quad \hat{\boldsymbol{\sigma}} = -\frac{\partial L}{\partial\nabla_0\otimes\mathbf{u}} \equiv \hat{\boldsymbol{\sigma}}(\boldsymbol{\upsilon}, \nabla_0\otimes\mathbf{u}) \tag{1.10}$$

The Hamilton functional (1.1), after changing to the Lagrange function, can be written in the form

$$\begin{aligned} &F^*[\mathbf{u}, \nabla_0\otimes\mathbf{u}, \mathbf{v}, \mathbf{p}_{(2)}] = \\ &= \int_{\tau_1}^{\tau_2}\left\{\int_{X_0}\left[\mathbf{p}\cdot\mathbf{v} - L(\mathbf{v}, \nabla_0\otimes\mathbf{u}) + \mathbf{u}\cdot\left(\frac{\partial\mathbf{p}}{\partial\tau} - \mathbf{f}_0\right)\right]dV_0 - \int_{\partial X_0}\boldsymbol{\sigma}_n^+\cdot\mathbf{u}\,d\Sigma_0\right\}d\tau - \\ &- \int_{X_0}\mathbf{u}_{(2)}^*\cdot\mathbf{p}_{(2)}dV_0 \end{aligned} \tag{1.11}$$

For the variation of this functional, taking into account the equations of state (1.10), we obtain

$$\begin{aligned} \delta F^*[\mathbf{u}, \mathbf{p}_{(2)}] &= \int_{\tau_1}^{\tau_2}\left\{\int_{X_0}\left(\frac{\partial\mathbf{p}}{\partial\tau} - \mathbf{f}_0 - \nabla_0\cdot\hat{\boldsymbol{\sigma}}\right)\cdot\delta\mathbf{u}\,dV_0 + \int_{\partial X_0}(\boldsymbol{\sigma}_n - \boldsymbol{\sigma}_n^+)\cdot\delta\mathbf{u}\,d\Sigma_0\right\}d\tau + \\ &+ \int_{X_0}[(\mathbf{u}_{(2)} - \mathbf{u}_{(2)}^*)\cdot\delta\mathbf{p}_{(2)} - \mathbf{u}_{(1)}\cdot\delta\mathbf{p}_{(1)}]dV_0 \end{aligned} \tag{1.12}$$

By equating to zero the variation of the functional F^*, as a necessary condition for an extremum, we obtain the equations of the locally formulated boundary-value problem in displacements

$$\nabla_0\cdot\hat{\boldsymbol{\sigma}}\left(\frac{\partial\mathbf{u}}{\partial\tau}, \nabla_0\otimes\mathbf{u}\right) + \mathbf{f}_0 = \frac{\partial}{\partial\tau}\left(\mathbf{p}\left(\frac{\partial\mathbf{u}}{\partial\tau}, \nabla_0\otimes\mathbf{u}\right)\right) \tag{1.13}$$

$$\boldsymbol{\sigma}_n\big|_{\partial X_0} = \boldsymbol{\sigma}_n^+, \quad \mathbf{u}\big|_{\tau=\tau_1} = 0, \quad \mathbf{u}\big|_{\tau=\tau_2} = \mathbf{u}_{(2)}^* \tag{1.14}$$

For the Hamilton function we assume that

$$\begin{aligned} &H(\mathbf{p}, \nabla_0\otimes\mathbf{u}) = W(\mathbf{p}) + U_0(\nabla_0\otimes\mathbf{u}) \\ &W(\mathbf{p}) = \frac{1}{2\rho}\mathbf{p}\cdot\mathbf{p}, \quad \rho = \frac{\delta m}{\delta V_0} = \rho^*\frac{\delta V^*}{\delta V_0}, \quad \rho^* = \frac{\delta m}{\delta V^*} \end{aligned}$$

where ρ is the density, i.e. the mass δm of a physically small element, referred to its volume δV_0 in the initial state; in the linear approximation $\rho_* \approx \rho(1-e)$, $e = \nabla_0\cdot\mathbf{u}$.

The Lagrange function and the equations of state will then take the form

$$L(\boldsymbol{v}, \nabla_0 \otimes \mathbf{u}) = \frac{\rho}{2}\boldsymbol{v} \cdot \boldsymbol{v} - U_0(\nabla_0 \otimes \mathbf{u})$$

$$\mathbf{p} = \rho \boldsymbol{v}, \quad \hat{\sigma} = \frac{\partial U_0(\nabla_0 \otimes \mathbf{u})}{\partial \nabla_0 \otimes \mathbf{u}} \equiv \hat{\sigma}(\nabla_0 \otimes \mathbf{u})$$

Equation (1.3) can then be written in the form

$$\nabla_0 \cdot \hat{\sigma} + \mathbf{f}_0 = \rho \frac{\partial \boldsymbol{v}}{\partial \tau} \tag{1.15}$$

For elastic isotropic materials the deformation potential energy density U_0 is a function of seven independent scalar invariants of the tensors $\hat{e} = \frac{1}{2}(\nabla_0 \otimes \mathbf{u} + \mathbf{u} \otimes \nabla_0)$ and $\hat{c} = \frac{1}{2}(\nabla_0 \otimes \mathbf{u} - \mathbf{u} \otimes \nabla_0)$ [2]

$$\begin{aligned} U_0 &= U_0(I(\hat{e}), I(\hat{e}^2), I(\hat{c}^2), I(\hat{e}^3), I(\hat{c}^2 \cdot \hat{e}), I(\hat{c}^2 \cdot \hat{e}^2), I(\hat{c}^2 \cdot \hat{e} \cdot \hat{c} \cdot \hat{e}^2)) \equiv \\ &\equiv U_0(A_1, A_2, \ldots, A_7) \end{aligned} \tag{1.16}$$

where

$$\hat{c}^i \cdot \hat{e}^j = \underbrace{\hat{c} \cdot \hat{c} \cdot \ldots \cdot \hat{c}}_{i} \cdot \underbrace{\hat{e} \cdot \hat{e} \cdot \ldots \cdot \hat{e}}_{j},$$

where $I(\cdot)$ is the trace of the second-rank tensor.

Taking formulae (1.16) into account, we can write the equation of state for the stress tensor of the non-linear theory of elasticity

$$\hat{\sigma} \equiv \frac{\partial U_0}{\partial(\nabla_0 \otimes \mathbf{u})} = \frac{\partial U_0}{\partial A_i}\frac{\partial A_i}{\partial(\nabla_0 \otimes \mathbf{u})}$$

2. THE QUASI-STATIC FORMULATION OF BOUNDARY-VALUE PROBLEMS FOR THIN PLATES. EXISTENCE AND UNIQUENESS CONDITIONS

Suppose the elastic body is a thin plate of limited dimensions. In the initial state the plate is characterized by the middle surface ∂X_0^c and the thickness $2h$. The position of an arbitrary point of the plate is defined by the radius vector $\mathbf{r}_0 = \mathbf{r}_{0*} + \mathbf{r}_{03}$, where $\mathbf{r}_{0*}$ is the radius vector of points in the middle surface of the plate ($\xi^3 = 0$); $\mathbf{r}_{03} = \xi^3 \boldsymbol{\ni}_0^3$ is the vector of the position of points along the normal to the middle surface ($\xi^3 \in [-h, h]$) and $\boldsymbol{\ni}_0^3$ is the unit vector in the direction of the normal to the middle surface.

For quasi-static loading

$$\int_{X_0} \mathbf{f}_0 dV_0 + \int_{\partial X_0} \boldsymbol{\sigma}_n^+ d\Sigma_0 = 0$$

boundary-value problem (1.15), (1.14) consists of the equation of equilibrium and the boundary condition on the surface ∂X_0

$$\nabla_0 \cdot \hat{\sigma} + \mathbf{f}_0 = 0, \quad \boldsymbol{\sigma}_n\big|_{\partial X_0} = \boldsymbol{\sigma}_n^+ \tag{2.1}$$

Here the Hamilton functional (1.1) is identical with the Lagrange functional

$$J[\mathbf{u}] = \int_{X_0} [U_0(\nabla_0 \otimes \mathbf{u}) - \mathbf{f}_0 \cdot \mathbf{u}] dV_0 - \int_{\partial X_0} \boldsymbol{\sigma}_n^+ \cdot \mathbf{u} d\Sigma_0 \tag{2.2}$$

The sufficient conditions for convexity of the functional (2.2) are similar to conditions (1.8) and can be written in the form

$$\int\limits_{X_0} \delta^2 U_0(\mathbf{V}_0 \otimes \mathbf{u})dV_0 > 0, \quad \int\limits_{X_0} (\mathbf{V}_0 \cdot \delta\hat{\sigma}) \cdot \delta\mathbf{u}dV_0 \geq 0 \tag{2.3}$$

To investigate the first condition of (2.3) we will use expression (1.16) for the potential U_0 as a function of the scalar invariance of the tensors $\hat{e}$ and $\hat{c}$. In this case the first Gâteaux differential of the functional $J[\mathbf{u}]$ in the direction of the vector φ can be written as follows:

$$J'(\mathbf{u}, \boldsymbol{\varphi}) = \int\limits_{X_0} \left[\frac{\partial U_0(A_1, \ldots, A_7)}{\partial A_i} \left(\frac{d}{d\theta} A_i(\mathbf{V}_0 \otimes \mathbf{u} + \theta \mathbf{V}_0 \otimes \boldsymbol{\varphi}) \right) \Bigg|_{\theta = 0} - \mathbf{f}_0 \cdot \boldsymbol{\varphi} \right] dV_0 - \\ - \int\limits_{\partial X_0} \boldsymbol{\sigma}_n^+ \cdot \boldsymbol{\varphi} d\Sigma_0, \quad \theta \in (0, 1), \quad i = 1, \ldots, 7 \tag{2.4}$$

Here and henceforth summation is carried out over repeated subscripts.

The second Gâteaux differential in the directions φ and ψ when $\psi = \varphi$ can be represented in the form

$$J''(\mathbf{u}, \boldsymbol{\varphi}, \boldsymbol{\varphi}) = \int\limits_{X_0} \left[\frac{\partial^2 U_0(A_1, \ldots, A_7)}{\partial A_i \partial A_j} \left(\frac{d}{d\theta} A_i(\mathbf{V}_0 \otimes \mathbf{u} + \theta \mathbf{V}_0 \otimes \boldsymbol{\varphi}) \right) \Bigg|_{\theta = 0} + \\ + \frac{\partial U_0(A_1, \ldots, A_7)}{\partial A_i} \frac{\partial}{\partial A_j} \left(\left(\frac{d}{d\theta} A_i(\mathbf{V}_0 \otimes \mathbf{u} + \theta \mathbf{V}_0 \otimes \boldsymbol{\varphi}) \right) \Bigg|_{\theta = 0} \right) \right] \times \\ \times \left(\frac{d}{d\gamma} A_j(\mathbf{V}_0 \otimes \mathbf{u} + \gamma \mathbf{V}_0 \otimes \boldsymbol{\varphi}) \right) \Bigg|_{\gamma = 0} dV_0 \equiv \int\limits_{X_0} \frac{\partial^2 U_0}{\partial q^{mn} \partial q^{ts}} \left(\frac{\partial \varphi_m}{\partial \xi^n} \frac{\partial \varphi_t}{\partial \xi^s} \right) dV_0 \\ \theta, \gamma \in (0, 1); \quad m, n, t, s = 1, 2, 3; \quad i, j = 1, \ldots, 7; \quad q^{mn} = \frac{\partial u_n}{\partial \xi^m}, \quad q^{ts} = \frac{\partial u_s}{\partial \xi^t}. \tag{2.5}$$

Hence, by expression (2.5) the local sufficient condition for convexity of the functional is that the following quadratic form should be positive definite

$$\frac{\partial^2 U_0}{\partial q^{mn} \partial q^{ts}} \left(\frac{\partial \varphi_m}{\partial \xi^n} \frac{\partial \varphi_t}{\partial \xi^s} \right) > 0, \quad \forall \boldsymbol{\varphi} \in V; \quad m, n, t, s = 1, 2, 3 \tag{2.6}$$

According to Sylvester's criterion the sign definiteness of the quadratic form of inequality (2.6) is equivalent to the condition that all nine principal minors $\Delta_1, \ldots, \Delta_9$ of quadratic form (2.6) should be positive.

We will assume that, in the range of variation of the external load considered, the second condition of (2.3) is satisfied. In this case the following theorem of the existence and uniqueness of a minimum of the functional $J[\mathbf{u}]$ holds [3, 4].

Theorem 1. Suppose Y is a weakly closed subspace of a reflexive Banach space W, while the functional $J[\mathbf{u}]$ is twice continuously Gâteaux differentiable. Then

A. If, for all $\mathbf{u}, \varphi \in Y$ the first Gâteaux differential $J'[\mathbf{u}, \varphi]$ of the functional $J[\mathbf{u}]$ in the direction φ is linear and continuous with respect to φ, while the second Gâteaux differential $J''[\mathbf{u}, \varphi, \psi]$ satisfies the condition $J''[\mathbf{u}, \varphi, \varphi] \geq 0$, a minimum of the functional $J[\mathbf{u}]$ exists in the space Y.

B. If a minimum of the functional exists and, moreover, the functional $J[\mathbf{u}]$ is strictly convex, this minimum is unique in the space Y.

The proof requires the construction of a solution of boundary-value problem (2.1) in the form of a weakly converging minimizing sequence. A version of the construction of such a solution for thin plates is proposed below.

3. A MATHEMATICAL MODEL OF THE THEORY OF PLATES

The sufficient conditions for existence and uniqueness. The Lagrange functional (2.2) of the variational formulation of boundary-value problem (2.1) is considered in a reflexive Banach space of the generalized functions

$$W = \left\{ \mathbf{u} \in W_2^1(X_0) \times W_2^1(X_0) \times W_2^1(X_0) \colon \frac{l}{V_0} \int_{X_0} \mathbf{u}\, dV_0 = \boldsymbol{\alpha},\ \frac{1}{V_0} \int_{X_0} \nabla_0 \times \mathbf{u}\, dV_0 = \boldsymbol{\beta} \right\} \tag{3.1}$$

Here $W_2^1(X_0) \times W_2^1(X_0) \times W_2^1(X_0)$ is a Sobolev space with norm

$$\|\mathbf{u}\|_{[W_2^1(X_0)]^3} = \left\{ \frac{1}{V_0} \left[\int_{X_0} \frac{1}{l^2}(u_1^2 + u_2^2 + u_3^2)\, dV_0 + \int_{X_0} \sum_{i,j=1}^{3} \left(\frac{\partial u_i}{\partial \xi^j}\right)^2 dV_0 \right] \right\}^{1/2}$$

$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are specified constant vectors, and l is the characteristic dimension of the middle surface of the plate.

We take the following sequence of vector functions as the minimizing functions for the functional $J[\mathbf{u}]$

$$\mathbf{u}_m = \sum_{i=1}^{m} \left(\frac{\mathbf{r}_{03}}{l}\right)^{(i-1)} \overset{i-1}{\cdot} \hat{u}^{(i)}(\mathbf{r}_{0*}), \quad m \in \mathbf{N}; \quad \left(\frac{\mathbf{r}_{03}}{l}\right)^{(i-1)} \equiv \frac{1}{l^{i-1}} \underbrace{\mathbf{r}_{03} \otimes \ldots \otimes \mathbf{r}_{03}}_{i-1} \tag{3.2}$$

Here $\left\{ \left(\frac{\mathbf{r}_{03}}{l}\right)^{(i-1)} \right\}$ $(i \in \mathbf{N})$ is a specified basis of tensors of increasing valency, the superscripts $(i-1)$ and (i) indicate the valency of the tensor functions and $\mathbf{N}$ is the set of natural numbers.

In this case the Lagrange functional (2.2) for the function $\mathbf{u}_m$ and its first variation can be represented in the form

$$J[\mathbf{u}_m] = \int_{\partial X_0^c} \left[\tilde{U}_0 - \sum_{i=1}^{m} \hat{F}^{(i)} \overset{i}{\cdot} \hat{u}^{(i)} \right] d\Sigma_0^c + \oint_s \sum_{i=1}^{m} \hat{Q}_{21}^{(i)+} \overset{i}{\cdot} \hat{u}^{(i)} dl_0$$

$$\delta J[\mathbf{u}_m] = \int_{\partial X_0^c} \sum_{i=1}^{m} \left(\boldsymbol{\ni}_0^3 \cdot \hat{Q}_{22}^{(i+1)} - \nabla_{0*} \cdot \hat{Q}_{21}^{(i+1)} - \hat{F}^{(i)} \right) \overset{i}{\cdot} \delta\hat{u}^{(i)} \delta\Sigma_0^c +$$

$$+ \oint_s \sum_{i=1}^{m} (\mathbf{n} \cdot \hat{Q}_{21}^{(i+1)} - \hat{Q}_{21}^{(i)+}) \cdot \delta\hat{u}^{(i)} dl_0^i \tag{3.3}$$

$$\tilde{U}_0 = \int_{-h}^{h} U_0 d\xi^3, \quad \hat{F}^{(i)} = \int_{-h}^{h} \mathbf{f}_0 \otimes \left(\frac{\mathbf{r}_{03}}{l}\right)^{(i-1)} d\xi^3 + \boldsymbol{\sigma}_{3+} \otimes \left(\frac{\mathbf{r}_{03}}{l}\right)_+^{(i-1)} - \boldsymbol{\sigma}_{3-} \otimes \left(\frac{\mathbf{r}_{03}}{l}\right)_-^{(i-1)}$$

$$\boldsymbol{\sigma}_3 = \boldsymbol{\ni}_0^3 \cdot \hat{\sigma}, \quad \nabla_{0*} \equiv \frac{\partial}{\partial \mathbf{r}_{0*}} \quad \hat{Q}_{21}^{(i+1)} = \int_{-h}^{h} \hat{\sigma} \otimes \left(\frac{\mathbf{r}_{30}}{l}\right)^{(i-1)} d\xi^3,$$

$$\hat{Q}_{22}^{(i+1)} = \int_{-h}^{h} \hat{\sigma} \otimes \frac{\partial}{\partial \xi^3} \left(\frac{\mathbf{r}_{30}}{l}\right)^{(i-1)} d\xi^3, \quad \hat{Q}_{21}^{(i)+} = \int_{-h}^{h} \boldsymbol{\sigma}_n^+ \otimes \left(\frac{\mathbf{r}_{30}}{l}\right)^{(i-1)} d\xi^3$$

The boundary values of the corresponding quantities on the upper and lower bases of the plate are denoted by plus and minus subscripts, $\hat{e}_1^{(i+1)} = \nabla_{0*} \otimes \hat{u}^{(i)}$ and $\hat{e}_2^{(i+1)} = \boldsymbol{\ni}_0^3 \otimes \hat{u}^{(i)}$ are generalized coordinates, $\hat{Q}_{21}^{(i+1)}$ and $\hat{Q}_{22}^{(i+1)}$ are generalized forces, conjugate to them, s is the closed contour of the middle surface of the plate and dl_0 is an element of arc.

From the conditions for a minimum of the functional (3.3) in the space W we obtain the boundary-value problem, formulated in terms of the generalized coordinates and generalized forces introduced

$$\nabla_0 \cdot \hat{Q}_{21}^{(i+1)} - \mathfrak{I}_0^3 \cdot \hat{Q}_{22}^{(i+1)} + \hat{F}^{(i)} = 0; \quad [\mathbf{n} \cdot \hat{Q}_{21}^{(i+1)}]_{\partial X_0^c} = \hat{Q}_{21}^{(i)+}; \quad i = 1, \dots, m \tag{3.4}$$

The conditions for convexity of the functional (3.3) follow from conditions (2.3) and can be represented in the form

$$\int_{\partial X_0^c} [\delta \hat{Q}_{21}^{(i+1)} \overset{i+1}{\cdot\cdot} \delta \hat{e}_1^{(i+1)} + \delta \hat{Q}_{22}^{(i+1)} \overset{i+1}{\cdot\cdot} \delta \hat{e}_2^{(i+1)}] d\Sigma_0^c > 0$$

$$\int_{\partial X_0^c} \Bigg[\delta \hat{Q}_{21}^{(i+1)} \overset{i+1}{\cdot\cdot} \delta \hat{e}_1^{(i+1)} - \delta \hat{Q}_{22}^{(i+1)} \overset{i+1}{\cdot\cdot} \delta \hat{e}_2^{(i+1)} + \tag{3.5}$$

$$+ \left(\delta \boldsymbol{\sigma}_{3+} \otimes \left(\frac{\mathbf{r}_{03}}{l}\right)_+^{(i-1)} - \delta \boldsymbol{\sigma}_{3-} \otimes \left(\frac{\mathbf{r}_{03}}{l}\right)_-^{(i-1)}\right) \overset{i}{\cdot\cdot} \delta \hat{u}^{(i)}\Bigg] d\Sigma_0^c > 0$$

The results obtained enable us to formulate the following theorem on the sufficient local conditions for the existence and uniqueness of a weak minimum of the Lagrange functional.

Theorem 2. A. Suppose

(1) in a weakly closed subspace W of the reflexive Banach space $[W_2^1(X_0)]^3$ the Lagrange functional $J[\mathbf{u}]$ is twice continuously Gâteaux differentiable;

(2) in the space of the functions W a unique solution $\mathbf{u}^1$ of the linear boundary-value problem of the theory of elasticity, corresponding to (2.1), exists;

(3) for the tensors $\hat{\sigma}^*$ and $\hat{\sigma}^l$ ($\hat{\sigma}^* \equiv \hat{\sigma} - \hat{\sigma}^l$, $\hat{\sigma}^l$ is the stress tensor of the linear theory of elasticity) the following condition is satisfied

$$\left|\int_{X_0} \hat{\sigma}^*(\mathbf{u}) \cdot\cdot (\nabla_0 \otimes \boldsymbol{\varphi})^T dV_0\right| \le \left|\int_{X_0} (\hat{\sigma}^l(\mathbf{u}) \cdot\cdot (\nabla_0 \otimes \boldsymbol{\varphi})^T dV_0)\right| \quad \text{for all} \quad \mathbf{u}, \boldsymbol{\varphi} \in W$$

(4) the coefficients

$$[(\mathfrak{I}_0^3)^{j-1} \overset{j-1}{\cdot\cdot} \hat{f}_0^{(j)}] \cdot \mathfrak{I}_0^i, \quad [(\mathfrak{I}_0^3)^{j-1} \overset{j-1}{\cdot\cdot} \hat{\sigma}^{+(j)}] \cdot \mathfrak{I}_0^i$$

of the functions

$$\mathbf{f}_0 = \sum_{j=1}^{\infty} \left(\frac{\mathbf{r}_{03}}{l}\right)^{(j-1)} \overset{j-1}{\cdot\cdot} \hat{f}_0^{(j)}(\mathbf{r}_{0*}), \quad \boldsymbol{\sigma}_n^+ = \sum_{j=1}^{\infty} \left(\frac{\mathbf{r}_{03}}{l}\right)^{(j-1)} \overset{j-1}{\cdot\cdot} \hat{\sigma}_n^{+(j)}(\mathbf{r}_{0*})$$

belong to the space $L_2(X_0^c)(j \in \mathbf{N}, i = 1, 2, 3)$;

(5) the following sufficient conditions for the convexity of the Lagrange functional are satisfied

$$\Delta_1, \Delta_2, \dots, \Delta_9 > 0 \quad \text{for all} \quad \boldsymbol{\varphi} \ne 0, \quad \forall \mathbf{u} \in W$$

$$\int_{X_0} (\nabla_0 \cdot \delta\hat{\sigma}) \cdot \delta\mathbf{u} dV_0 > 0 \quad \text{for all} \quad \mathbf{u} \in W$$

Then a minimum of the functional $J[\mathbf{u}]$ exists in the space of the functions W.

B. Suppose the above-mentioned conditions are satisfied and the functional $J[\mathbf{u}]$ is strictly convex. Then, the minimum of the functional is unique in the space W.

Proof. A. Existence. We will take as the minimizing functions the sequence of functions $\mathbf{u}_m \in W$, for which

$$J[\mathbf{u}_m] \to \inf_{\mathbf{u} \in W} J[\mathbf{u}] \quad \text{as} \quad m \to \infty$$

For the functions of the external force loads, for which conditions 4 are satisfied, the sequence $\mathbf{u}_m$ is weakly convergent as $m \to \infty$ to the function

$$\tilde{\mathbf{u}} = \sum_{i=1}^{\infty} \left(\frac{\mathbf{r}_{03}}{l}\right)^{(i-1)} {}^{i-1}_{\cdot}\hat{u}^{(i)}(\mathbf{r}_{0*})$$

in the space W [5].

It follows from the condition for the weak closure of W in the space $[W_2^1(X_0)]^3$ that the boundary element $\mathbf{u}$ of the sequence $\mathbf{u}_m$ belongs to W.

Satisfaction of condition 3 of the theorem for the tensors $\hat{\sigma}^*$, $\hat{\sigma}^l$ ensures continuity of the first Gâteaux differential of the functional $J[\mathbf{u}]$. From these conditions and the conditions of convexity of the Lagrange functional, there follows the semi-continuity from below of $J[\mathbf{u}]$, and the following inequality [4] is satisfied

$$J[\tilde{\mathbf{u}}] \leq \underline{\lim} J[\mathbf{u}_m]$$

Hence $J[\tilde{\mathbf{u}}] = \inf_{\upsilon_m \in W} J[\mathbf{u}_m]$, and the function $\mathbf{u}$ realises a minimum of the functional $J[\mathbf{u}]$ in the space W.

B. Uniqueness. The strict convexity of the functional is ensured by the satisfaction of conditions 5. Since a minimum of $J[\mathbf{u}]$ exists, this minimum is unique.

If follows from the use of the variational formulation of problem (2.1) that the conditions of the theorem are sufficient for the existence and uniqueness of a weak solution of the boundary-value problem.

The results obtained here can be used, for example, to formulate and solve corresponding extremal problems of the non-linear theory of elasticity for thin plates and shells [6].

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